Math 122 Midterm 2 solutions

Question 1 (True or False) - 2 points each

1 T
2 F
3 F
4 T
5 T
6 F
7 T
8 F

Question 2 - 15 points

a) Sylow Theorems:
Given a group G of order $p^n m$
1) There exists a Sylow p-subgroup.
2) All Sylow p-subgroups are conjugates.
3) The number of Sylow p-subgroups divides m and congruent to 1 (mod p).

b) Consider a group of $35 = 5.7$ elements. By the Sylow theorems, there is exactly one Sylow 5-subgroup and another Sylow 7-subgroup, both are normal subgroups of G. An element of G can only have order 1, 5, 7 or 35 (divisors of 35):
There is 1 element of order 1
There are 6 elements of order 7
There are 4 elements of order 5, thus there must be 35-(1+4+6)=24 elements of order 35. Each of these elements generate a 35 element cyclic sub group, which is the whole group we are considering.

Question 3 - 8 points

a) Define $T = \{s \in S | G_s = G\}$
b) The order of the set S is the sum of the disjoint orbits, thus by Stabilizer-Orbit equation:

$$\#S = \sum_{orbits} \#O_s = \sum_{orbits} \frac{\#G}{\#G_s} = \sum_{s \in T} \frac{\#G}{\#G} + \sum_{orbits\ s \notin T} \frac{\#G}{\#G_s} = \#T + pA, A \geq 1$$

Question 4 - 16 points

a) A matrix A is orthogonal $\Leftrightarrow A^t A = I$.
b) $T : V \to V$ is an orthogonal operator on an inner product space $V \Leftrightarrow < Tv, Tw > = < v, w >$.
c) Let c be an eigenvector of T, such that $Tv = cv$, then apply condition in b), $< Tv, Tv > = < v, v >$

But by bilinearity of inner product, $< Tv, Tv >=< cv, cv >= c^2 v$, thus $c^2 = 1 \rightarrow c = \pm 1$.

Question 5 - 15 points

a) Define $D_n =< g, h : g^n = h^2 = e, hgh^{-1} = g^{-1} >$, or equivalently, $D_n = \{g^i h^j | g^n = h^2 = e, gh = hg^{-1}\}$. D_n is abelian if and only if $n \leq 2$

b) S_4 has order $24 = 2^3 3$, thus there is 1 Sylow 2-subgroup, of order 8. To show that this subgroup is isomorphic to D_8, need to find an element of order 4 $(= g)$, another element of order 2 $(= h)$, such that they are related by $hgh^{-1} = g^{-1}$.

An element of order 4 in S_4 is a 4-cycle: $g = (1234)$, consider $g^{-1} = (4321)$, thus our element h has to switch 1 and 4, 2 and 3, or $h = (1\,4)(2\,3), h$ has order 2. Hence the subgroup of S_4 generated by $g = (1\,2\,3\,4), h = (1\,4)(2\,3)$ is isomorphic to D_8.